Estimation of Reliability in the Two-Parameter Geometric Distribution

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Abstract

In this article, the reliabilities $R(t) = P(X \geq t)$, when X follows two-parameter geometric distribution and $R = P(X \leq Y)$, arises under stress-strength setup, when X and Y assumed to follow two-parameter geometric independently have been found out. Maximum Likelihood Estimator (MLE) and an Unbiased Estimator (UE) of these have been derived. MLE and UE of the reliability of k-out-of-m system have also been derived. The estimators have been compared through simulation study.

AMS Subject Classification: 62N05; 62F10.

Key Words: k-out-of-m system; Maximum Likelihood Estimator; Stress-strength reliability; Unbiased estimator.

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1 Introduction

Various lifetime models have been proposed to represent lifetime data. Most of these models assume lifetime to be a continuous random variable. However, it is sometimes impossible or inconvenient to measure the life length of a device on a continuous scale. In practice, we come across situations where lifetimes are recorded on a discrete scale. Discrete life distributions have been mentioned by Barlow and Proschan [1]. Here one may consider lifetime to be the number of successful cycles or operations of a device before failure. For example, the bulb in xerox machine lights up each time a copy is taken. A spring may breakdown after completing a certain number of cycles of 'to-and-fro' movements.

The study of discrete distributions in lifetime models is not very old. Yakub and Khan [2] considered the geometric distribution as a failure law in life testing and obtained various parametric and nonparametric estimation procedures for reliability characteristics. Bhattacharya and Kumar [3] discussed the parametric as well as Bayesian approach to the estimation of the mean life cycle and reliability for this model for complete as well as censored sample. Krishna and Jain [4] obtained classical and Bayes estimation of reliability for some basic system configurations. Modeling in terms of two-parameter geometric and estimation of its parameters and related functions are of special interest to a manufacturer who wishes to offer a minimum warranty life cycle of the items produced.

The two-parameter geometric distribution [abbreviated as $Geo(r, \theta)$] given by

$$P(X = x) = (1 - \theta)\theta^{x-r} \;\; ; \; x = r, r+1, r+2, ... \;\; 0 < \theta < 1, \; r \in \{0, 1, 2, ...\}, \tag{1.1}$$

is the discrete analog of two-parameter exponential distribution. If X follows a two-parameter exponential distribution, $[X]$, the integer part of X, has a two-parameter geometric distribution [see Kalbfleish and Prentice [5, Ch. 3]]. The reliability of a component when X follows two-

parameter geometric distribution is given by

$$R(t) = \theta^{t-r}; \quad t = r, r+1, r+2, ... \tag{1.2}$$

Laurent [6] and Tate [7] obtained the uniformly minimum variance unbiased estimator (UMVUE) of the reliability function for the two-parameter exponential model. Different estimators of this reliability function have been discussed in Sinha [8].

If a system consists of m identical components each follows two-parameter geometric distribution, then the reliability of k-out-of-m system is given by

$$R_s(t) = P(X_{(m-k+1)} \geq t) = \sum_{i=k}^{m} \binom{m}{i} R(t)^i [1 - R(t)]^{m-i}; \quad t = r, r+1, r+2, ... \tag{1.3}$$

Special cases of $R_s(t)$ give series system (for $k = m$) and parallel system (for $k = 1$).

In the stress-strength setup, $R = P(X \leq Y)$ originated in the context of the reliability of a component of strength Y subjected to a stress X. The component fails if at any time the applied stress is greater than its strength and there is no failure when $X \leq Y$. Thus R is a measure of the reliability of the component. Many authors considered the problem of estimation of R in continuous setup in the past. Particularly, for the two-parameter exponential set up, Beg [9] derived the MLE and the UMVUE of R. In the discrete setup, the reference list is very limited. Maiti [10] has considered stress (or demand) X and strength (or supply) Y as independently distributed geometric random variables, whereas Sathe and Dixit [11] assumed as negative binomial variables, and derived both MLE and UMVUE of R. Maiti and Kanji [12] has derived some expressions of R using a characterization and Maiti [13, 14] considered MLE, UMVUE and Bayes Estimation of R for some discrete distributions useful in life testing. All the above mentioned works have been concentrated on one-parameter family of discrete distributions.

If X and Y follow two-parameter geometric distributions with parameters (θ_1, r_1) and (θ_2, r_2) respectively, then

$$\begin{aligned} R &= \rho\theta_2^{\delta} \qquad \text{for } \delta > 0 \\ &= 1-(1-\rho)\theta_1^{-\delta} \;\; \text{for } \delta < 0, \end{aligned} \tag{1.4}$$

where $\rho = \frac{1-\theta_1}{1-\theta_1\theta_2}$ and $\delta = r_1 - r_2$.

Here we are interested to see whether similar estimators are obtained in case of the two-parameter geometric distribution, the discrete analog of the two-parameter exponential distribution. Then, it might be straightforward to use the two-parameter geometric distribution in the discrete life testing problem where a minimum warranty life cycle of the item is offered. In this article, we have found out some estimators of both $R(t)$ and $R_s(t)$ for complete as well as censored sample. Some estimators of R have also been provided. The estimators have been compared through simulation study.

The paper is organized as follows. In section 2, we have derived MLE and UE of both $R(t)$ and $R_s(t)$. We have also derived MLE of these reliability functions for type-I censored sample. MLE and an unbiased estimator of R have been found out in section 3. In section 4, simulation results have been reported. Section 5 concludes.

2 Estimation of $R(t)$ and $R_s(t)$

Let $(X_1, X_2, ..., X_n)$ be a random sample from $Geo(r, \theta)$ and $(X_{(1)}, X_{(2)}, ..., X_{(n)})$ be ordered sample. Maximum Likelihood Estimator of r and θ are $X_{(1)}$ and $\frac{S}{n+S}$ respectively, where

$S=\sum_{i=1}^{n}\left(X_i-X_{(1)}\right)=\sum_{i=1}^{n}\left(X_{(i)}-X_{(1)}\right)$. ML Estimators of $R(t)$ and $R_s(t)$ are given by

$$\begin{aligned}\hat{R}_M(t) &= 1 \qquad \text{for } t\le X_{(1)}\\ &= \left[\frac{S}{n+S}\right]^{t-X_{(1)}} \quad \text{for } t>X_{(1)}\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\hat{R}_{sM}(t) &= 1 \qquad \text{for } t\le X_{(1)}\\ &= \sum_{i=k}^{m}\binom{m}{i}\left[\hat{R}_M(t)\right]^i\left[1-\hat{R}_M(t)\right]^{m-i} \quad \text{for } t>X_{(1)}\end{aligned} \tag{2.6}$$

respectively.

Suppose we record the observations $(X_{(1)},X_{(2)},...,X_{(p)}),p\le n$ that are failed before a pre-specified number of cycles c and remainings survive beyond c. Then, MLE of r and θ are $X_{(1)}$ and $\frac{S^*}{p+S^*}$ respectively, where $S^*=\sum_{i=1}^{p}\left(X_{(i)}-X_{(1)}\right)+(n-p)\{(c+1)-X_{(1)}\}$. Hence ML Estimators of $R(t)$ and $R_s(t)$ are given by

$$\begin{aligned}\hat{R^*}_M(t) &= 1 \qquad \text{for } t\le X_{(1)}\\ &= \left[\frac{S^*}{p+S^*}\right]^{t-X_{(1)}} \quad \text{for } t>X_{(1)}\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\hat{R^*}_{sM}(t) &= 1 \qquad \text{for } t\le X_{(1)}\\ &= \sum_{i=k}^{m}\binom{m}{i}\left[\hat{R^*}_M(t)\right]^i\left[1-\hat{R^*}_M(t)\right]^{m-i} \quad \text{for } t>X_{(1)}\end{aligned} \tag{2.8}$$

respectively.

Theorem 2.1 $\left(X_{(1)},S\right)$ *is sufficient statistic for* (r,θ).

Proof: Let $\underline{X}=(X_1,X_2,...,X_n)$, we have to prove that $P(\underline{X}=\underline{x}|X_{(1)}=u,\ S=s)$ does not depend on r and θ.

Given $X_{(1)} = u$ and $S = s$, $\underline{X}$ is an n-dimensional random variable with domain $A_{u,s} = \{\underline{x} | x_{(1)} = u,\ \sum_{i=2}^{n} x_i = s + (n-1)u\}$.

For $\underline{x} \in A_{u,s}$,

$$P(\underline{X} = \underline{x} | X_{(1)} = u,\ S = s) = \frac{P(\underline{X} = \underline{x})}{\sum_{\underline{y} \in A_{u,s}} P(\underline{X} = \underline{y})}$$

and

$$P(\underline{X} = \underline{x}) = \prod_{i=1}^{n} P(X_i = x_i) = (1-\theta)^n \theta^{s+n(u-r)}.$$

Thus $P(\underline{X} = \underline{x} | X_{(1)} = u,\ S = s) = \frac{1}{|A_{u,s}|}$, where $|A_{u,s}|$ is the number of elements in $A_{u,s}$. The number of elements in $A_{u,s}$ is the number of possible n-uplets $(x_1, x_2, ..., x_n)$ such that $x_{(1)} = u$ and $\sum_{i=2}^{n} x_i = s + (n-1)u$ which clearly does not depend on θ and r.

But $\left(X_{(1)}, S\right)$ is not complete as it is to be seen from the following counter example.

Counterexample 2.1 *Let us define* $g(.,\ .)$ *as*

$$\begin{aligned} g\left(X_{(1)}, S\right) &= 1 \quad \textit{if } X_{(1)} = r+2,\ S = 0 \\ &= -1 \quad \textit{if } X_{(1)} = r+1,\ S = n \\ &= 0 \quad \textit{otherwise.} \end{aligned}$$

Now $X_{(1)}$ *and* S *can take values* $r+2$ *and* 0 *with the probability* $(1-\theta)^n\theta^{2n}$ *(for* $X_{(2)} = r+2,\ ...,\ X_{(n)} = r+2$*), and* $X_{(1)}$ *and* S *can take values* $r+1$ *and* n *with the probability* $(1-\theta)^n\theta^{2n}$ *(one such particular situation is* $X_{(2)} = r+2,\ ...,\ X_{(n-1)} = r+2,\ X_{(n)} = r+3$*). Therefore, it is found that* $E_{r,\ \theta}\left[g\left(X_{(1)}, S\right)\right] = 0$ *but* $g\left(X_{(1)}, S\right) \neq 0$.

The upcoming theorem will demonstrate the conditional distribution of X for given $\left(X_{(1)}, S\right)$.

Theorem 2.2 *The conditional distribution of* X *given* $\left(X_{(1)}, S\right)$ *is as following:*

For $n = 1$,

$$P\left(X = x / X_{(1)}, S\right) = 1$$

For $n = 2$,

$$
\begin{aligned}
P\left(X = x/X_{(1)}, S\right) &= \frac{1}{2} \quad \textit{if} \ \ x = X_{(1)} \\
&= \frac{1}{2} \quad \textit{if} \ \ x = X_{(1)} + S
\end{aligned}
$$

For $n \geq 3, \ S < n$,

$$
\begin{aligned}
P\left(X = x/X_{(1)}, S\right) &= \binom{S - \left(x - X_{(1)}\right) + n - 2}{S - \left(x - X_{(1)}\right)} \Big/ \binom{S + n - 1}{S} \quad \textit{if } X_{(1)} \leq x \leq X_{(1)} + S \\
&= 0 \qquad\qquad\qquad \textit{otherwise.}
\end{aligned}
$$

For $n \geq 3, \ S \geq n$,

$$
\begin{aligned}
P\left(X = x/X_{(1)}, S\right) &= \binom{S + n - 2}{S} \Big/ \left\{ \binom{S + n - 1}{S} - \binom{S - 1}{n - 1} \right\} \quad \textit{if } x = X_{(1)} \\
&= \left\{ \binom{S - \left(x - X_{(1)}\right) + n - 2}{S - \left(x - X_{(1)}\right)} - \binom{S - \left(x - X_{(1)}\right) - 1}{n - 2} \right\} \Big/ \left\{ \binom{S + n - 1}{S} - \binom{S - 1}{n - 1} \right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } X_{(1)} < x \leq X_{(1)} + S - (n - 1) \\
&= \binom{S - \left(x - X_{(1)}\right) + n - 2}{S - \left(x - X_{(1)}\right)} \Big/ \left\{ \binom{S + n - 1}{S} - \binom{S - 1}{n - 1} \right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } X_{(1)} + S - (n - 1) < x \leq X_{(1)} + S \\
&= 0 \qquad\qquad\qquad \textit{otherwise.}
\end{aligned}
$$

Proof: Joint distribution of $X_1, \ X_2, \ ..., \ X_n$ is given by

$$
P(X_1, \ X_2, \ ..., \ X_n) = (1 - \theta)^n \theta^{\sum_{i=1}^{n}(X_i - r)} = (1 - \theta)^n \theta^{S + n(X_{(1)} - r)}, \quad r \leq X_{(1)}.
$$

Now,

$$
\begin{aligned}
P\left(X = x/X_{(1)}, S\right) &= \frac{\sum_{X_2, \ X_3, \ ..., \ X_n} P(X = x, \ X_2, \ ..., \ X_n/X_{(1)}, \ S)}{\sum_{X_1, \ X_2, \ X_3, \ ..., \ X_n} P(X_1, \ X_2, \ ..., \ X_n/X_{(1)}, \ S)} \\
&= \frac{\sum_{(X_2, \ X_3, \ ..., \ X_n/X_{(1)}, \ S)} 1}{\sum_{(X_1, \ X_2, \ X_3, \ ..., \ X_n/X_{(1)}, \ S)} 1}.
\end{aligned}
$$

Here the denominator is equivalent to finding out the total number of ways in which S indistinguishable balls can be placed in n cells so that at least one cell remains empty. In general, if

there are r indistinguishable balls to be placed randomly in k cells, then the number of distinguishable distributions is $\binom{k+r-1}{r}$ whereas the number of distinguishable distributions in which no cells remains empty is $\binom{r-1}{k-1}$. Therefore, we get the denominator as $\binom{S+n-1}{S} - \binom{S-1}{n-1}$ if $S \geq n$ and if $S < n$, it will be $\binom{S+n-1}{S}$. Similarly, the numerator is equivalent to finding out the total number of ways in which $S - \left(x - X_{(1)}\right)$ indistinguishable balls can be placed in $n-1$ cells so that at least one cell remains empty and hence, we get it as $\binom{S-\left(x-X_{(1)}\right)+n-2}{S-\left(x-X_{(1)}\right)} - \binom{S-\left(x-X_{(1)}\right)-1}{n-2}$ if $S - \left(x - X_{(1)}\right) \geq n-1$ and if $S - \left(x - X_{(1)}\right) < n-1$, it will be $\binom{S-\left(x-X_{(1)}\right)+n-2}{S-\left(x-X_{(1)}\right)}$. Hence the theorem follows.

Since $\left(X_{(1)}, S\right)$ is sufficient but not complete statistic for (r, θ), we are handicapped of searching the UMVUE of any estimable function of these parameters using the Lehmann-Scheffé theorem. Hence, we will find an improved estimator of the reliability functions using the Rao-Blackwell theorem.

Define

$$
\begin{aligned}
Y &= 1 \qquad \text{if } \; X_1 \geq t \\
&= 0 \qquad \text{otherwise.}
\end{aligned}
$$

Then $R(t) = E(Y) = P\left(X_1 \geq t\right)$. Using the Rao-Blackwell theorem, an unbiased estimator of $R(t)$ is given as follows:

for $n = 1$,

$$
\begin{aligned}
\tilde{R}_U(t) &= 1 \qquad \text{if } t \leq X_{(1)} \\
&= 0 \qquad \text{if } \; t > X_{(1)}.
\end{aligned}
$$

for $n = 2$,

$$\begin{aligned}
\tilde{R}_U(t) &= 1 && \text{if } t \le X_{(1)} \\
&= \frac{1}{2} && \text{if } X_{(1)} < t \le X_{(1)} + S \\
&= 0 && \text{if } t > X_{(1)} + S.
\end{aligned}$$

for $n \geq 3$ and $S < n$,

$$\begin{aligned}
\tilde{R}_U(t) &= 1 \qquad \text{if } t \le X_{(1)} \\
&= \sum_{x=t}^{X_{(1)}+S} \binom{S-\big(x-X_{(1)}\big)+n-2}{S-\big(x-X_{(1)}\big)} \Big/ \binom{S+n-1}{S} \quad \text{if } X_{(1)} < t \le X_{(1)} + S \\
&= 0 \qquad\qquad \text{if } t > X_{(1)} + S.
\end{aligned}$$

It can also be written as

$$\begin{aligned}
\tilde{R}_U(t) &= 1 \qquad \text{if } t \le X_{(1)} \\
&= \sum_{x=t}^{X_{(1)}+S} \frac{n-1}{X_{(1)}+S+n-1} \prod_{j=1}^{n-2} \frac{\big(X_{(1)}+S+n-x-1-j\big)}{\big(X_{(1)}+S+n-1-j\big)} \quad \text{if } X_{(1)} < t \le X_{(1)} + S \\
&= 0 \qquad\qquad \text{if } t > X_{(1)} + S.
\end{aligned}$$

For $n \geq 3$, $S \geq n$,

$$\begin{aligned}
\tilde{R}_U(t) &= 1 \qquad\qquad \text{if } t \le X_{(1)} \\
&= \sum_{x=t}^{X_{(1)}+S-(n-1)} \left\{ \binom{S-\big(x-X_{(1)}\big)+n-2}{S-\big(x-X_{(1)}\big)} - \binom{S-\big(x-X_{(1)}\big)-1}{n-2} \right\} \Big/ \left\{ \binom{S+n-1}{S} - \binom{S-1}{n-1} \right\} \\
&\quad + \sum_{x=X_{(1)}+S-(n-1)+1}^{X_{(1)}+S} \binom{S-\big(x-X_{(1)}\big)+n-2}{S-\big(x-X_{(1)}\big)} \Big/ \left\{ \binom{S+n-1}{S} - \binom{S-1}{n-1} \right\} \\
&\qquad\qquad \text{if } X_{(1)} < t \le X_{(1)} + S - (n-1) \\
&= \sum_{x=t}^{X_{(1)}+S} \binom{S-\big(x-X_{(1)}\big)+n-2}{S-\big(x-X_{(1)}\big)} \Big/ \left\{ \binom{S+n-1}{S} - \binom{S-1}{n-1} \right\} \\
&\qquad\qquad \text{if } X_{(1)} + S - (n-1) < t \le X_{(1)} + S \\
&= 0 \qquad\qquad \text{otherwise.}
\end{aligned}$$

In other way the estimator $\tilde{R}_U(t)$ is to be UMVUE if it is uncorrelated with all unbiased estimator of zero. We take a class of unbiased estimator of zero as $U_0 = \{u : \sum_{i=1}^{n} c_i X_i = u,\ \sum_{i=1}^{n} c_i = 1\}$. If $\tilde{R}_U(t)$ is UMVUE, then $Cov(U_0,\ \tilde{R}_U(t)) = 0$ i.e. $Cov(1000.U_0,\ 1000.\tilde{R}_U(t)) = 0$. Analytical derivation seems to be intractable. We go for simulation study taking some particular choices of $(c_1,\ c_2,\ ...,\ c_n)$ and different t, and 1000 covariances have been calculated and their averages have been shown in Tables 7-8. It is noticed that they are not uncorrelated and hence $\tilde{R}_U(t)$ is not UMVUE.

The variance of this unbiased estimator will be smaller than the unbiased estimator $\frac{\sum_{i=1}^{n} I(X_i \geq t)}{n}$, where $I(.)$ is the indicator function.

To study the asymptotic behavior of $\tilde{R}_U(t)$ we conduct a simulation study taking different values of parameters. 10000 estimates of $\tilde{R}_U(t)$ and $\hat{R}_M(t)$, their variances, 95% confidence limits and coverage probability (CP) have been shown in table 9. Histogram of $\tilde{R}_U(t)$ for $n = 20, r = 15, t = 25, \theta = 0.96$ has been shown in Figure 1. In this set up the true reliability, $R(t) = 0.6648326$. The figure is near normal. From the table 9, it is also evident from coverage probability point of view, $\hat{R}_M(t)$ is better if $0.02 < R(t) < 0.5$, otherwise $\tilde{R}_U(t)$ is better. From the table it is observed that asymptotic variance is approximately $\frac{R(t)(1-R(t))}{2n}$.

Define

$$
\begin{aligned}
Z &= 1 && \text{if at least } k \text{ of } X_i\text{'s among } X_1,\ X_2,\ ...,\ X_m \text{ are greater than or equal to } t \\
&= 0 && \text{otherwise.}
\end{aligned}
$$

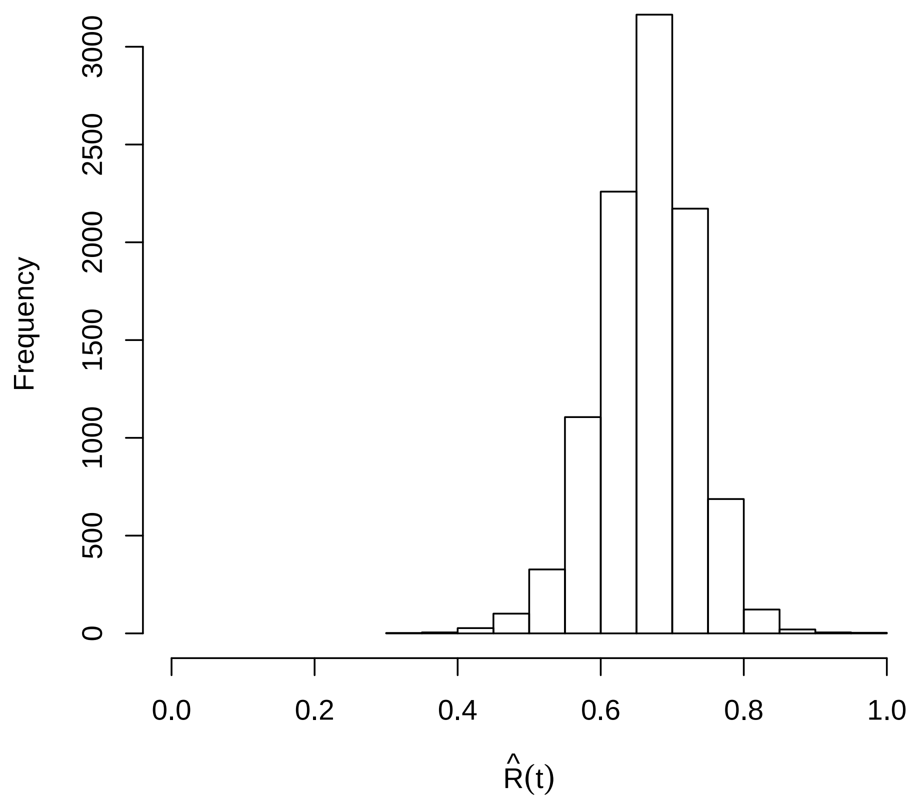

Figure 1: Histogram of $\tilde{R}_U(t)$.

Then $R_s(t) = E(Z) = \sum_{i=k}^{m} \binom{m}{i}[R(t)]^i[1-R(t)]^{m-i}$. Using the Rao-Blackwell theorem, an unbiased estimator of $R_s(t)$ is given by (for $2 \le m < n$)

$$\begin{aligned}
\tilde{R}_{sU}(t) &= 1 \qquad \text{if } t \le X_{(1)} \\
&= \sum_{i=k}^{m} \binom{m}{i} [\tilde{R}_U(t)]^i [1-\tilde{R}_U(t)]^{m-i} \quad \text{if } X_{(1)} < t \le X_{(1)} + S \\
&= 0 \qquad\qquad \text{if } \ t > X_{(1)} + S.
\end{aligned}$$

3 Estimation of R

Let $(X_1, X_2, ..., X_{n_1})$ and $(Y_1, Y_2, ..., Y_{n_2})$ be random samples from $Geo(r_1, \theta_1)$ and $Geo(r_2, \theta_2)$ respectively. $\left(X_{(1)}, S_1\right)$ and $\left(Y_{(1)}, S_2\right)$ are defined in the same way as in section 2. Hence ML Estimator of R is given by

$$\begin{aligned}\hat{R}_M &= \hat{\rho}\left(\frac{S_2}{n_2 + S_2}\right)^{\hat{\delta}} \qquad \text{for } \hat{\delta} > 0 \\ &= 1 - (1 - \hat{\rho})\left(\frac{S_1}{n_1 + S_1}\right)^{-\hat{\delta}} \quad \text{for } \hat{\delta} < 0,\end{aligned}$$

where $\hat{\rho} = \frac{n_1 n_2 + n_1 S_2}{n_1 n_2 + n_1 S_2 + n_2 S_1}$ and $\hat{\delta} = X_{(1)} - Y_{(1)}$.

We define censored scheme in the same way as in section 2, with pre-specified censored cycles c_1 and c_2 and with p_1 and p_2 censored observations.

Then, ML Estimator of R is given by

$$\begin{aligned}\hat{R^*}_M &= \hat{\rho^*}\left(\frac{S_2^*}{p_2 + S_2^*}\right)^{\hat{\delta}} \qquad \text{for } \hat{\delta} > 0 \\ &= 1 - (1 - \hat{\rho^*})\left(\frac{S_1^*}{p_1 + S_1^*}\right)^{-\hat{\delta}} \quad \text{for } \hat{\delta} < 0,\end{aligned}$$

where $\hat{\rho^*} = \frac{p_1 p_2 + p_1 S_2^*}{p_1 p_2 + p_1 S_2^* + p_2 S_1^*}$ and $\hat{\delta} = X_{(1)} - Y_{(1)}$.

Define

$$\begin{aligned}Z &= 1 \qquad \text{if } \; X_1 \leq Y_1 \\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

Then $R = E(Z) = P(X_1 \leq Y_1)$. Application of the Rao-Blackwell theorem gives an unbiased estimator of R as

$$\begin{aligned}
\tilde{R}_U &= \frac{1}{n_1} + \sum_{x=X_{(1)}+1}^{Y_{(1)}} f(x/X_{(1)}, S_1) + \sum_{x=Y_{(1)}}^{min(W_1,W_2)} \sum_{y=x}^{W_2} f(x/X_{(1)}, S_1) f(y/Y_{(1)}, S_2) \quad \text{if } X_{(1)} < Y_{(1)} \\
&= \frac{1}{n_1} + \sum_{x=X_{(1)}}^{min(W_1,W_2)} \sum_{y=x}^{W_2} f(x/X_{(1)}, S_1) f(y/Y_{(1)}, S_2) \quad \text{if } X_{(1)} = Y_{(1)} \\
&= \frac{1}{n_1} \sum_{y=X_{(1)}}^{W_2} f(y/Y_{(1)}, S_2) + \sum_{x=X_{(1)}+1}^{min(W_1,W_2)} \sum_{y=x}^{W_2} f(x/X_{(1)}, S_1) f(y/Y_{(1)}, S_2) \quad \text{if } X_{(1)} > Y_{(1)}
\end{aligned}$$

where, $W_1 = X_{(1)} + S_1$, $W_2 = Y_{(1)} + S_2$. The variance of this unbiased estimator will be smaller than the unbiased estimator $\frac{1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} I(X_i < Y_j)$.

4 Simulation Study

4.1 Discussion on simulation results relating to $R(t)$ and $R_s(t)$.

In order to have an idea about the selection of an estimator between Maximum Likelihood Estimator (MLE) and Unbiased Estimator (UE), Mean Squared Errors (MSEs) and hence percent relative efficiency using these MSEs have been calculated for $R(t)$ and $R_s(t)$. We generate sample of size n and on the basis of this sample, calculate MLE and UE. MLEs have been calculated for complete as well as censored (type-I defined in earlier section) samples. 10000 such estimates have been calculated and results, on the basis of these estimates have been reported in Tables $1-6$. Initial set up for parameters has been chosen as ($n = 20,\ r = 15,\ c = 25,\ \theta = 0.8,\ t = 25,\ k = 2,\ m = 8$). Each table has been prepared considering different choices of a particular parameter, keeping others fixed at initial set up. All simulations and calculations have been done using R-Software and algorithms used can be obtained by contacting the corresponding author. Different columns of a table are as follows:

Col.1: Component Reliability, $R(t)$

Col.2: Average of MLEs of $R(t)$ for complete sample

Col.3: Average of MLEs of $R(t)$ for censored sample

Col.4: Percent relative efficiency of MLE of $R(t)$ for complete sample to that of censored sample

Col.5: Average of UEs of $R(t)$

Col.6: Percent relative efficiency of UE to MLE of $R(t)$

Col.7: System Reliability, $R_s(t)$

Col.8: Average of MLEs of $R_s(t)$ for complete sample

Col.9: Average of MLEs of $R_s(t)$ for censored sample

Col.10: Percent relative efficiency of MLE of $R_s(t)$ for complete sample to that of censored sample

Col.11: Average of UEs of $R_s(t)$

Col.12: Percent relative efficiency of UE to MLE of $R_s(t)$

For estimation of component as well as system reliability, MLE performs quite well from efficiency point of view except for very less reliable component whose importance in practice is not so much meaningful (Table 1 and Table 2, Col.6 and Col.12). Even though the MLE is biased, the combined effect of variance and bias is less than the variance of the proposed UE. If sample size n increases, as expected, MLE and UE for component as well as system reliability become equally efficient (Table 4, Col.6 and Col.12). If censored number of cycle c increases, MLEs for complete as well as censored sample become equally efficient (Table 3, Col.4 and Col.10).

4.2 Discussion on simulation results relating to R.

In order to have an idea about the nature of the MLEs under complete sample and censored sample, we have calculated percent relative efficiency of $\hat{R}_M$ with respect to $\hat{R^*}_M$ i.e. $\frac{\text{MSE of } \hat{R^*}_M \times 100}{\text{MSE of } \hat{R}_M}$ and presented in tables for $r_1 = 5,\ 10$, $r_2 = 10,\ 5$, $\theta_1 = 0.7,\ 0.8$, $\theta_2 = 0.7,\ 0.8$ and for different values of c_1 and c_2 (Table 10-13). Here, we take $n_1 = n_2 = 10$, and 1000 estimates of R have been taken for calculating MSEs. We observe that, as expected, MLE of R for censored sample approaches to that of complete sample as both c_1 and c_2 increase. Fixing any one of c_1 and c_2, and increasing the remaining, efficiency of MLE for censored sample sometimes increases but there is no guarantee.

In order to have an idea about the selection of an estimator between UE and MLE, we have calculated MSEs of Estimates of R. We prepared tables (Table14-18) for (i) $\theta_1 = 0.1,\ \theta_2 = 0.1$, (ii) $\theta_1 = 0.5,\ \theta_2 = 0.5$, (iii) $\theta_1 = 0.8,\ \theta_2 = 0.2$, (iv) $\theta_1 = 0.9,\ \theta_2 = 0.9$, (v) $\theta_1 = 0.2,\ \theta_2 = 0.8$,and r_1 and r_2 equal to 5, 10, 15 and 20, $n_1 = n_2 = 10$. In these tables, values in 1st row indicates true value of R, 2nd and 3rd rows indicate average of estimate of MLE and UE of R, and 4th and 5th rows indicate MSEs of MLE and UE of R respectively.

We observe that in almost all cases, MLE is better that UE for R in mean square error sense. Therefore, as soon as we entered to unbiased class, we are losing some efficiency. It is to be noticed that MLE in this case is not an unbiased estimator. Moreover, MLE has a computational ease.

5 Concluding Remark

This paper takes into account the inferential aspects of reliability with the two-parameter geometric lifetime. The continuous distributions are widely referenced probability laws used

in reliability and life testing for continuous data. When the lives of some equipments and components are being measured by the number of completed cycles of operations or strokes, or in case of periodic monitoring of continuous data, the discrete distribution is a natural choice. At the same time, if a minimum warranty life cycle of the items are provided, the two-parameter geometric distribution is the simplest but an important choice. Under this set up estimators of reliability functions- under mission time as well as under stress-strength set up, have been viewed. It is interesting to note that, unlike the case of the two-parameter exponential, the estimators of the parameters of the two-parameter geometric distribution are not complete. As a result, we only get unbiased estimators of the reliability functions for the two-parameter geometric set up.

In most of the situations, MLE gives better result than the UE in mean square sense. As soon as we entered to unbiased class, we are loosing some efficiency. If one is ready to sacrifice the unbiased criteria of the estimator, the MLE in this case is preferable. It is to be noticed here that MLE is not an unbiased estimator. Moreover, MLE has a computational ease.

References

[1] R. E. Barlow and F. Proschan, *Mathematical Theory of Reliability*, John Willy & Sons, Inc., New York, 1967.

[2] M. Yakub and A. H. Khan, *Geometric failure law in life testing*, Pure and Applied Mathematika Science 14(1981), pp. 69-76.

[3] S. K. Bhattacharya and S. Kumar, *Discrete life testing*, IAPQR Transactions 13(1985), pp. 71-76.

[4] H. Krishna and N. Jain, *Classical and Bayes Estimation of Reliability Characteristics of some Basic System Configurations with Geometric Lifetimes of Components*, IAPQR Transactions 27(2002), pp. 35-49.

[5] J. D. Kalbfleish and R. L. Prentice, *The Statistical Analysis of Failure Time Data*, Wiley, New York, 1980.

[6] A. G. Laurent, *Conditional distribution of order statistics and distribution of the reduced ith order statistics of the exponential model*, Ann. Math. Statist. 34(1963), pp. 652-657.

[7] R. F. Tate, *Unbiased estimation: Functions of location and scale parameters*, Ann. Math. Statist. 30(1959), pp. 341-366.

[8] S. K. Sinha, *Reliability and Life Testing*, Wiley Eastern Limited, New Delhi, 1986.

[9] M. A. Beg, *On the estimation of $P(Y < X)$ for the two-parameter exponential distribution*, Metrika, 27(1980), pp. 29-34.

[10] S. S. Maiti, *Estimation of $P(X \leq Y)$ in the geometric case*, Journal of the Indian Statistical Association 33(1995), pp. 87-91.

[11] Y. S. Sathe and U. J. Dixit, *Estimation of $P[X \leq Y]$ in the negative binomial distribution*, Journal of Statistical Planning and Inference 93(2001), pp. 83-92.

[12] S. S. Maiti, and A. Kanji, *A note on Characterization of $P(X \leq Y)$ for Discrete Life Distributions*, Journal of Applied Statistical Science 14(2005), pp. 275-279.

[13] S. S. Maiti, *Bayesian Estimation of $P(X \leq Y)$ for some Discrete Life Distributions*, Contributions to Applied and Mathematical Statistics 3(2005), pp. 81-96.

[14] S. S. Maiti, *On Estimation of $P(X \leq Y)$ for Discrete Distributions useful in Life Testing*, IAPQR Transactions 31(2006), pp. 39-46.

Table 1: Calculations relating to $R(t)$ and $R_s(t)$

$n = 20,\ r = 15,\ c = 25,\ \theta = 0.8,\ k = 2,\ m = 8$

t	Col.1	Col.2	Col.3	Col.4	Col.5	Col.6	Col.7	Col.8	Col.9	Col.10	Col.11	Col.12
16	0.8	0.79393	0.79491	104.89	0.80022	113.64	0.99992	0.999708	0.999706	100.15	0.99977	138.10
17	0.64	0.62917	0.63338	106.57	0.63867	119.58	0.99571	0.982063	0.982062	100.31	0.98545	128.81
18	0.512	0.49571	0.50086	106.73	0.51097	109.54	0.96979	0.928259	0.929258	99.76	0.93871	120.28
19	0.4096	0.39881	0.40459	110.55	0.41452	98.88	0.90330	0.836803	0.838851	100.28	0.85352	111.01
20	0.32768	0.31945	0.32699	113.34	0.33407	92.92	0.79549	0.722729	0.729116	101.38	0.74399	102.54
25	0.10737	0.10622	0.11228	131.55	0.10647	80.50	0.20909	0.220599	0.236196	125.40	0.22473	83.46
30	0.03518	0.04018	0.04422	151.20	0.03661	83.31	0.03009	0.056895	0.067862	168.06	0.05466	83.63
31	0.02814	0.03461	0.03810	151.25	0.03011	89.96	0.01981	0.044496	0.054024	178.83	0.04043	91.94
35	0.01153	0.01476	0.01754	212.54	0.01108	104.80	0.00355	0.011759	0.017823	334.81	0.00923	101.20
40	0.00378	0.00692	0.00868	210.03	0.00473	122.46	0.00039	0.003788	0.006659	375.59	0.00271	107.04
45	0.00123	0.00271	0.00342	226.31	0.00157	174.54	4.26×10^{-5}	0.000858	0.001667	737.01	0.00048	189.46

Table 2: Calculations relating to $R(t)$ and $R_s(t)$

$n = 20,\ r = 15,\ c = 25,\ \theta = 0.8,\ t = 25,\ m = 8$

k	Col.1	Col.2	Col.3	Col.4	Col.5	Col.6	Col.7	Col.8	Col.9	Col.10	Col.11	Col.12
1	0.10737	0.10841	0.11121	116.64	0.11827	78.010	0.59695	0.56534	0.57109	105.37	0.59365	94.259
3	0.10737	0.10821	0.11118	118.38	0.11805	78.110	0.45796	0.06392	0.06981	137.86	0.07894	63.170
6	0.10737	0.10829	0.11124	119.96	0.11813	78.048	3.53×10^{-05}	0.000263	0.000408	374.98	0.000436	39.115
8	0.10737	0.10933	0.11229	118.16	0.11929	77.648	1.76×10^{-08}	7.12×10^{-07}	1.51×10^{-06}	1081.50	1.46×10^{-06}	25.299

Table 3: Calculations relating to $R(t)$ and $R_s(t)$

$n = 20,\ r = 15,\ \theta = 0.8,\ t = 25,\ k = 2,\ m = 8$

c	Col.1	Col.2	Col.3	Col.4	Col.5	Col.6	Col.7	Col.8	Col.9	Col.10	Col.11	Col.12
20	0.10737	0.10790	0.11599	164.86	0.11772	78.104	0.20909	0.21959	0.24290	160.05	0.24860	76.145
25	0.10737	0.10819	0.11123	118.83	0.11801	78.133	0.20909	0.22043	0.22924	118.50	0.24941	76.362
30	0.10737	0.10829	0.10946	107.18	0.11814	78.029	0.20909	0.22059	0.22397	107.18	0.24964	76.268
35	0.10737	0.10899	0.10962	103.95	0.11892	77.747	0.20909	0.22251	0.22432	103.93	0.25179	76.101
40	0.10737	0.10842	0.10866	101.60	0.11828	78.084	0.20909	0.22093	0.22160	101.54	0.24995	76.446
45	0.10737	0.10772	0.10780	100.63	0.11751	78.329	0.20909	0.21914	0.21935	100.58	0.24801	76.463

Table 4: Calculations relating to $R(t)$ and $R_s(t)$

$$r = 15,\ c = 25,\ \theta = 0.8,\ t = 25,\ k = 2,\ m = 8$$

n	Col.1	Col.2	Col.3	Col.4	Col.5	Col.6	Col.7	Col.8	Col.9	Col.10	Col.11	Col.12
10	0.10737	0.10599	0.11160	127.07	0.12610	62.804	0.20909	0.21957	0.23384	122.73	0.27625	63.291
15	0.10737	0.10769	0.11140	120.18	0.12087	72.608	0.20909	0.22115	0.23150	118.81	0.25938	71.594
20	0.10737	0.10909	0.11298	118.70	0.11984	77.429	0.20909	0.22491	0.23418	118.39	0.25443	75.910
25	0.10737	0.10819	0.11075	117.64	0.11599	81.963	0.20909	0.21916	0.22675	118.07	0.24232	79.985
50	0.10737	0.10830	0.10965	114.09	0.11215	90.156	0.20909	0.21598	0.22022	114.83	0.22757	88.475
100	0.10737	0.10761	0.10823	111.36	0.10949	95.064	0.20909	0.21199	0.21403	111.92	0.21772	93.938
200	0.10737	0.10735	0.10769	110.46	0.10829	97.569	0.20909	0.21011	0.21124	110.76	0.21297	96.925

Table 5: Calculations relating to $R(t)$ and $R_s(t)$

$$n = 20,\ c = 25,\ \theta = 0.8,\ t = 25,\ k = 2,\ m = 8$$

r	Col.1	Col.2	Col.3	Col.4	Col.5	Col.6	Col.7	Col.8	Col.9	Col.10	Col.11	Col.12
0	0.00377	0.00552	0.00556	101.87	0.00471	101.81	0.00039	0.00182	0.00185	94.865	0.00162	88.625
5	0.01152	0.01420	0.01436	106.003	0.01366	86.929	0.00355	0.00885	0.00914	113.05	0.00904	79.946
10	0.03518	0.03885	0.03976	112.85	0.04073	78.387	0.03009	0.04675	0.04916	120.39	0.05214	72.434
15	0.10737	0.10779	0.11066	119.41	0.11756	78.399	0.20909	0.21937	0.22761	118.72	0.24815	76.659
20	0.32768	0.31915	0.32740	138.77	0.34411	89.196	0.79548	0.75086	0.75481	117.19	0.79071	121.971

Table 6: Calculations relating to $R(t)$ and $R_s(t)$

$$n = 20,\ r = 15,\ c = 25,\ t = 25,\ k = 2,\ m = 8.$$

θ	Col.1	Col.2	Col.3	Col.4	Col.5	Col.6	Col.7	Col.8	Col.9	Col.10	Col.11	Col.12
0.4	0.00010	0.00032	0.00032	101.01	0.00012	337.78	3.07×10^{-7}	1.49×10^{-5}	1.50×10^{-5}	107.31	4.39×10^{-6}	491.22
0.5	0.00097	0.00183	0.00184	103.22	0.00099	204.17	2.65×10^{-5}	0.00028	0.00029	120.32	0.00013	230.03
0.6	0.00605	0.00823	0.00831	104.67	0.00597	136.80	0.00099	0.00367	0.00367	111.03	0.00236	162.16
0.7	0.02824	0.03160	0.03213	110.33	0.02798	102.49	0.01995	0.03348	0.03482	118.42	0.02875	112.80
0.8	0.10737	0.10815	0.11123	118.94	0.10703	90.47	0.20909	0.22019	0.22923	119.02	0.21801	92.17
0.9	0.34867	0.34157	0.34866	141.31	0.34973	97.45	0.82891	0.78709	0.78698	120.64	0.79895	105.99
0.93	0.48398	0.47655	0.48212	165.12	0.48399	105.66	0.95725	0.93552	0.92884	155.35	0.94084	117.34
0.96	0.66483	0.66788	0.66198	198.66	0.66456	118.19	0.99731	0.99469	0.99086	355.23	0.99487	120.60
0.99	0.90438	0.93525	0.90791	173.29	0.90454	119.21	0.99999	0.99999	0.99996	390.01	0.99999	121.53

Table 7: Unbiased Estimator

$$n = 10,\ r = 15,\ c = 25,\ \theta = 0.8,\ k = 2,\ m = 8$$

Combination | t	20	25	30	35	40
$+1-1+1-1+1-1+1-1+1-1$	334142.2	30824.96	8892.633	13735.52	−9741.06
$+1+1+1+1+1-1-1-1-1-1$	37026.8	2746.153	87121.23	−23797.94	4468.285
$+1+1\ 0\ 0\ 0\ 0\ 0\ 0-1-1$	−78456.49	−31337.49	37854.66	843.4048	−17.16015
$+1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0-1$	50794.91	−36774.63	14240.32	1158.73	−2478.892
Reliability	0.32768	0.10737	0.03518	0.01152	0.00378

Table 8: Unbiased Estimator

$$n = 10,\ r = 15,\ c = 25,\ t = 25,\ k = 2,\ m = 8$$

Combination $\downarrow$ | $\theta \rightarrow$	0.4	0.5	0.6	0.7	0.8	0.9	0.93	0.96	0.99
$+1-1+1-1+1-1+1-1+1-1$	277.34	1297.87	−7990.63	−25294.98	29730.69	466268.7	−626848.4	1314555	2910383
$+1+1+1+1+1-1-1-1-1-1$	128.73	603.68	2339.78	2376.59	−12487.09	−378417.4	−551749.3	5398500	−532512.9
$+1+1\ 0\ 0\ 0\ 0\ 0\ 0-1-1$	78.22	−118.91	3483.52	21081.09	−60072.55	−107717.3	−605752.3	1757503	−13004957
$+1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0-1$	132.79	34.93	−941.95	3206.99	−5265.09	120432.4	−695584.4	530686.8	−9375230
Reliability	0.000105	0.000977	0.006047	0.028248	0.107374	0.348678	0.483982	0.664833	0.904382

Table 9: Calculation of Confidence Interval and CP of $R(t)$

$$n = 20,\ r = 15,\ c = 25,\ t = 25,\ Repetation = 10000$$

θ	Reliability	Unbiased Estimator				MLE			
		Mean	Variance	(LCL, UCL)	CP	Mean	MSE	LCL, UCL	CP
0.4	0.000104	0.000123	1.524×10^{-7}	(0.000642, 0.000888)	0.9698	0.000309	4.746×10^{-7}	(0.001041, 0.001659)	0.9862
0.5	0.000977	0.001016	4.575×10^{-6}	(0.003176, 0.005208)	0.9636	0.001860	8.889×10^{-6}	(0.003983, 0.007704)	0.9830
0.6	0.006046	0.005932	5.819×10^{-5}	(0.009019, 0.020883)	0.9450	0.008182	7.887×10^{-5}	(0.009225, 0.025589)	0.9701
0.7	0.028247	0.028009	0.000543	(0.017681, 0.073699)	0.9500	0.031625	0.000556	(0.014591, 0.077841)	0.9559
0.8	0.107374	0.107834	0.002879	(0.002658, 0.213009)	0.9599	0.108912	0.002606	(0.008854, 0.208971)	0.9523
0.9	0.348678	0.349326	0.006850	(0.187109, 0.511543)	0.9520	0.341149	0.006688	(0.180857, 0.501442)	0.9487
0.93	0.483837	0.483837	0.006413	(0.326884, 0.640790)	0.9518	0.476420	0.006768	(0.315177, 0.636989)	0.9569
0.96	0.664832	0.665289	0.004414	(0.535067, 0.795511)	0.9490	0.668659	0.005214	(0.527134, 0.810185)	0.9626
0.99	0.904382	0.904440	0.002129	(0.813998, 0.994883)	0.8560	0.935051	0.002511	(0.836834, 1.000000)	0.9739

Table 10: Efficiency of $\hat{R}_M$ with respect to $\hat{R}^*_M$

$r_1 = 10,\ r_2 = 5,\ \theta_1 = 0.7,\ \theta_2 = 0.8,\ R = 0.8543644$

$c_1 \mid c_2$	15	20	25	30
10	170.3204	168.5565	162.8126	184.5733
15	123.3550	119.4340	121.8824	124.4574
20	107.4592	108.5157	110.6393	108.1562
25	106.2693	104.6287	105.2644	103.2766

Table 11: Efficiency of $\hat{R}_M$ with respect to $\hat{R}^*_M$

$r_1 = 5,\ r_2 = 10,\ \theta_1 = 0.8,\ \theta_2 = 0.7,\ R = 0.8212655$

$c_1 \mid c_2$	15	20	25	30
10	147.774	154.9791	157.9302	162.8920
15	114.8463	118.9883	118.0901	121.116
20	110.571	109.2363	106.1009	111.0485
25	103.5806	104.4578	100.1903	103.9806

Table 12: Efficiency of $\hat{R}_M$ with respect to $\hat{R}^*_M$

$r_1 = 10,\ r_2 = 5,\ \theta_1 = 0.7,\ \theta_2 = 0.8,\ R = 0.2234182$

$c_1 \mid c_2$	10	15	20	25
15	147.8276	113.0031	108.1288	103.5886
20	159.4114	115.1075	105.5036	104.5615
25	150.5815	117.2885	104.9115	102.4807
30	140.7334	117.5558	107.1203	102.5214

Table 13: Efficiency of $\hat{R}_M$ with respect to $\hat{R}^*_M$

$r_1 = 10,\ r_2 = 5,\ \theta_1 = 0.8,\ \theta_2 = 0.7,\ R = 0.07639545$

$c_1 \mid c_2$	10	15	20	25
15	133.3005	109.2876	104.7854	103.7044
20	140.5820	108.7528	101.8761	100.2199
25	144.2547	107.1663	100.6334	101.4030
30	146.9855	114.6641	102.8547	101.3404

Table 14: MSEs of Estimator R

$r_2 \mid r_1$	5	10	15	20
	0.909091	9.09091×10^{-6}	9.09091×10^{-11}	9.09091×10^{-16}
	0.920557	0.000162	1.37222×10^{-7}	1.55348×10^{-9}
5	0.914516	0.343349	0.362465	0.344545
	0.005747	4.09776×10^{-7}	8.34178×10^{-11}	3.25312×10^{-16}
	0.006513	0.280415	0.300813	0.284422
	0.999999	0.909091	9.09091×10^{-6}	9.09091×10^{-11}
	0.9999468	0.918347	0.000160	2.19983×10^{-7}
10	0.999996	0.91221	0.350020	0.347819
	4.47059×10^{-08}	0.005862	8.72553×10^{-7}	1.81692×10^{-12}
	6.68094×10^{-11}	0.006658	0.288780	0.283411
	1	.999999	.909091	9.09091×10^{6}
	1	0.999964	0.916321	0.000191
15	1	0.999998	0.909904	0.331967
	2.63535×10^{-13}	2.19439×10^{-8}	0.005932	4.68203×10^{-7}
	8.26455×10^{-23}	2.72526×10^{-11}	0.006783	0.271694
	1	1	0.999999	0.909091
	1	0.999999	0.999964	.914222
20	1	1	0.999998	0.907743
	1.05866×10^{-18}	9.20165×10^{-13}	2.28706×10^{-8}	0.006325
	1.77863×10^{-32}	8.26455×10^{-23}	2.72526×10^{-11}	0.007260

Table 15: MSEs of Estimator R

$r_2 \vert r_1$	5	10	15	20
5	0.666667	0.020833	0.000651	2.03450×10^{-5}
	0.679177	0.026838	0.002103	0.000206
	0.671748	0.021386	0.001522	0.001859
	0.012088	0.000737	2.33284×10^{-5}	5.11005×10^{-7}
	0.012973	0.000982	3.90682×10^{-5}	0.000701
10	0.989583	0.666667	0.020833	0.000651
	0.984516	0.668551	0.026998	0.001960
	0.989622	0.660862	0.021230	0.001143
	0.000364	0.011225	0.000801	1.61857×10^{-5}
	0.002692	0.012219	0.000720	2.13649×10^{-5}
15	0.999674	0.989583	0.666667	0.020833
	0.998673	0.984826	0.669140	0.0265948
	0.999563	0.990105	0.661459	0.020817
	1.50795×10^{-5}	0.000312	0.012265	0.000715
	6.954222×10^{-6}	0.000227	0.013318	0.000845
20	0.999989	0.999674	0.989583	0.666667
	0.999867	0.998716	0.984689	0.670912
	0.999988	0.999662	0.989835	0.663109
	2.96125×10^{-7}	8.48119×10^{-7}	0.000345	0.012157
	2.92646×10^{-8}	1.94865×10^{-6}	0.000254	0.013208

Table 16: MSEs of Estimator R

$r_2 \vert r_1$	5	10	15	20
5	0.238095	7.61904×10^{-5}	2.43809×10^{-8}	7.80190×10^{-12}
	0.239828	0.000384	3.17275×10^{-6}	7.44942×10^{-8}
	0.242251	0.028820	0.031522	0.029636
	0.009815	1.03719×10^{-6}	4.49468×10^{-10}	6.71449×10^{-13}
	0.007733	0.005539	0.006393	0.005711
10	0.750339	0.238095	7.61905×10^{-5}	2.43809×10^{-8}
	0.753009	0.236090	0.003744	3.40036×10^{-6}
	0.742105	0.239565	0.033643	0.032150
	0.010630	0.008706	9.65263×10^{-7}	4.16401×10^{-10}
	0.012344	0.006546	0.006595	0.006113
15	0.918191	0.750339	0.238095	7.61905×10^{-5}
	0.918237	0.756132	0.240675	0.000352
	0.920495	0.746061	0.243698	0.032421
	0.003296	0.012589	0.009720	8.85057×10^{-7}
	0.004096	0.014265	0.007625	0.006236
20	0.973193	0.918191	0.750339	0.238095
	0.968532	0.919645	0.767043	0.240174
	0.973493	0.922001	0.757469	0.242044
	0.000989	0.003383	0.011126	0.009538
	0.001103	0.004192	0.012577	0.007625

Table 17: MSEs of Estimator R

$r_2 \| r_1$	5	10	15	20
	0.526316	0.310784	0.183515	0.108364
	0.526810	0.299460	0.174113	0.103238
5	0.526508	0.308696	0.182257	0.105726
	0.017799	0.012224	0.007922	0.0043331
	0.015585	0.011825	0.008655	0.005117
	0.720294	0.526316	0.310784	0.183515
	0.726634	0.529795	0.305464	0.175418
10	0.716990	0.529132	0.314470	0.183407
	0.011833	0.016580	0.013347	0.008355
	0.011819	0.014537	0.012886	0.009171
	0.834836	0.720294	0.526316	0.313784
	0.840993	0.722299	0.530707	0.305661
15	0.833705	0.712836	0.529979	0.314475
	0.007389	0.012591	0.015473	0.013614
	0.008308	0.012569	0.013530	0.013235
	0.902473	0.834836	0.720294	0.526316
	0.903279	0.840593	0.726930	0.525431
20	0.900223	0.833145	0.717270	0.525064
	0.003855	0.006583	0.012184	0.017255
	0.004660	0.007377	0.012206	0.015104

Table 18: MSEs of Estimator R

$r_2 \mid r_1$	5	10	15	20
5	0.952381 0.954791 0.953994 0.001314 0.001318	0.321076 0.300429 0.313513 0.012567 0.013776	0.102261 0.107198 0.107466 0.005071 0.006272	0.033509 0.038337 0.033587 0.001269 0.001443
10	0.999985 0.999870 0.999989 1.49867×10^{-7} 6.16543×10^{-9}	0.952381 0.952055 0.951173 0.001352 0.001373	0.312076 0.296181 0.309150 0.012463 0.013512	0.102261 0.103630 0.103415 0.004663 0.005783
15	1 0.999998 1 1.01544×10^{-10} 2.37772×10^{-17}	.999985 0.999865 0.999983 2.91485×10^{-7} 4.87421×10^{-8}	.952381 0.952267 0.951499 0.001478 0.001480	.312076 0.302442 0.315536 0.012921 0.014082
20	1 1 1 4.40539×10^{-13} 2.43492×10^{-24}	1 .999998 1 1.37763×10^{-10} 2.37772×10^{-17}	0.999985 0.999858 0.999986 1.86115×10^{-7} 8.33376×10^{-9}	0.952381 0.952217 0.951531 0.001436 0.001436

Table 19: Calculation of Confidence Interval and CP of R

$$n_1 = n_2 = 10,\ \theta_1 = \theta_2 = 0.9,\ Repetation = 10000$$

$(r_1,\ r_2)$	Reliability	Unbiased Estimator				MLE			
		Mean	Variance	(LCL, UCL)	CP	Mean	MSE	LCL, UCL	CP
(5, 20)	0.108363	0.108382	0.005508	(0.037085, 0.253851)	0.9540	0.104635	0.004648	(0.028995, 0.238266)	0.9372
(5, 15)	0.183515	0.184696	0.009011	(0.001353, 0.370744)	0.9650	0.176514	0.008193	(0.000888, 0.353917)	0.9510
(5, 10)	0.310784	0.309366	0.012371	(0.091370, 0.527362)	0.9540	0.300321	0.012822	(0.078386, 0.522256)	0.9592
(5, 5)	0.526316	0.524415	0.014289	(0.290129, 0.758701)	0.9504	0.524652	0.016296	(0.274403, 0.774854)	0.9627
(10, 5)	0.720294	0.721299	0.011505	(0.511075, 0.931525)	0.9545	0.730976	0.011572	(0.520138, 0.941814)	0.9560
(15, 5)	0.834836	0.832709	0.008217	(0.672409, 1.000000)	0.9629	0.840145	0.007324	(0.672409, 1.000000)	0.9470
(20, 5)	0.902472	0.902443	0.004755	(0.767295, 1.000000)	0.9545	0.905403	0.003967	(0.781953, 1.028853)	0.9545